Northwestern University

MATH 300-0 Exam 2
Spring Quarter 2024
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Instructions

- This examination consists of 9 pages, not including this cover page. Verify that your copy of this examination contains all 9 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 6 questions for a total of 58 points.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- You may tear off the reference sheet and blank pages at the end of the booklet, but you must hand them in at the end of the exam.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Let $f : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ be a function such that for all $[n] \in \mathbb{Z}/8\mathbb{Z}$, $f([n]) = [n^2]$.

   Note: No explanation is necessary unless otherwise specified.

   (a) (2 points) What is $f([5])$?

   Solution: $f([5]) = [25] = [1]$

   (b) (2 points) What is $f^{-1}([1])$, the preimage of $[1]$? Give your answer as a set of equivalence classes.

   Solution: The preimage of $[1]$ is the set $\{[1], [3], [5], [7]\}$

   (c) (2 points) What is the image of $f \circ f$? Give your answer as a set of equivalence classes.

   Solution: $\{[0], [1]\}$

   (d) (3 points) For what values of $k \in \{0, 1, \dots, 7\}$ is it true that $f([k]) = [k]$?

   Solution: Only for $k = 0, 1$.

2. (10 points) Suppose $f : A \to B$ is an injective function and $A_1, A_2 \subseteq A$. Prove that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$

Solution:

We show two set inclusions to prove this result.

($\subseteq$) Let $b \in f(A_1 \cap A_2)$ be arbitrary. Then there exists some $a \in A_1 \cap A_2$ such that $f(a) = b$. Since $a \in A_1$, we know $f(A_1)$ contains b. Likewise since $a \in A_2$, we know $f(A_2)$ contains b. Thus $b \in f(A_1) \cap f(A_2)$.

($\supseteq$) Let $b \in f(A_1) \cap f(A_2)$ be arbitrary. Then there exists $a_1 \in A_1$ and $a_2 \in A_2$ such that $f(a_1) = b$ and $f(a_2) = b$. Since f is injective, this implies $a_1 = a_2$. This element $a_1 \in A_2$ as well, so $a \in A_1 \cap A_2$. Then we know $f(A_1 \cap A_2)$ contains b.

3. Determine whether each relation below is an equivalence relation, and prove your claim.

(a) (7 points) Let R be a relation on the set $\mathbb{Z}$, where for any $n, m \in \mathbb{Z}$, we define $n \, R \, m$ to mean $\gcd(n, m) = 1$, i.e. the greatest common divisor of n and m is 1.

Solution: R is not an equivalence relation. To see this, note that R is not reflexive. If n is some integer greater than 1, then $\text{gdc}(n, n) = n \neq 1$. Since R isn't reflexive, it can't be an equivalence relation.

(b) (7 points) Let $\sim$ be a relation on $\mathbb{R}^3$ where for vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$, we define $\vec{u} \sim \vec{v}$ to mean that the first coordinate of $\vec{u}$ equals the first coordinate of $\vec{v}$. In other words, if $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then $\vec{u} \sim \vec{v}$ if and only if $u_1 = v_1$.

Solution:

$\sim$ is reflexive: It is clear that $\vec{u} \sim \vec{u}$ because $u_1 = u_1$.

$\sim$ is symmetric: If $\vec{u} \sim \vec{v}$ then $u_1 = v_1$. Thus $v_1 = u_1$ so $\vec{v} \sim \vec{u}$.

$\sim$ is transitive: If $\vec{u} \sim \vec{v}$ and $\vec{v} \sim \vec{w}$, this means $u_1 = v_1$ and $v_1 = w_1$. Then we know that $u_1 = w_1$ so $\vec{u} \sim \vec{w}$.

Because $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

(c) (3 points) For any relations above that are equivalence relations, describe the equivalence classes generated by that relation.

Solution: The equivalence classes of $\sim$ from part (b) form planes in $\mathbb{R}^3$, where each plane contains all vectors with the same first coordinate.

4. (5 points) Prove that set of all finite sequences of 1's and 0's is countable. For example, 1110 is a finite sequence because its length is 4, but 10101010... is not a finite sequence.

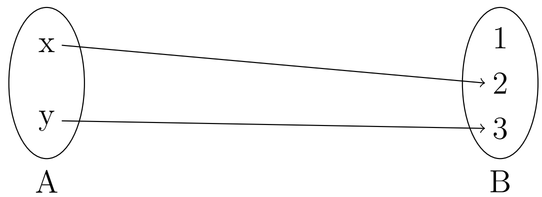
Solution: Let S be the set of all finite sequences of 1's and 0's.

Let A_i be the set of all sequences of 1's and 0's of length i, ie $A_1 = \{0, 1\}$, $A_2 = \{00, 11, 01, 10\}$, etc. Then the set of all finite sequences of 1's and 0's is the union of all A_i for $i = 1, 2, 3, \ldots$. In other words,

$$S = \bigcup_{i \in \mathbb{N} \setminus \{0\}} A_i$$

which is the countable union of countable (actually finite) sets. By Lemma C.2, S is countable.

5. Let $f : A \to B$ be a function given by the function diagram below.



(a) (7 points) Construct a left inverse $g : B \to A$ by explicitly stating the image of each element of B, i.e. by specifying $g(1)$, $g(2)$, and $g(3)$. Then prove that g is a left inverse.

Solution: Note: more than one answer is possible because $g(1) = y$ is also possible.

For all $b \in B$, define $g : B \to A$ by $g(1) = g(2) = x$ and $g(3) = y$.
To see that g is a left inverse, we show that $g \circ f = \mathrm{Id}_A$.

$$(g \circ f)(x) = g(f(x)) = g(2) = x$$

$$(g \circ f)(y) = g(f(y)) = g(3) = y$$

Thus $g \circ f = \mathrm{Id}_A$ and g is a left inverse of f.

(b) (5 points) If $h : B \to C$ is some surjective function, is $h \circ f$ always, sometimes, or never surjective? Explain your answer, using specific examples if necessary.

Solution:
$h \circ f$ is sometimes surjective.

Suppose h is the identity map on B. Then the image of $h \circ f$ does not include 1, so $h \circ f$ is not surjective.

On the other hand, suppose $C = \{\star\}$ and $h : B \to C$ is the constant map whose image is $\star$. Then $h \circ f$ is surjective.

6. (5 points) Prove that $[0,1]$ and $\mathbb{R}$ have the same cardinality by using the Schröder-Bernstein theorem.

Solution: $f : [0,1] \to \mathbb{R}$ given by $f(x) = x$ is clearly injective, so $|[0,1]| \leq |\mathbb{R}|$

$g : \mathbb{R} \to [0,1]$ given by $g(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\arctan(x)$ is an injective function whose image is $(0,1)$. Thus $|\mathbb{R}| \leq |[0,1]|$.

By the Schröder-Bernstein theorem, $|[0,1]| = |\mathbb{R}|$.

THERE IS NO EXAMINATION MATERIAL ON THIS PAGE.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

REFERENCE

You may cite any lemma listed below in your proofs.

Logic

Lemma L.1 (De Morgan's Laws for Logic) Let A, B be propositions. Then

$$\neg(A \wedge B) = \neg A \vee \neg B \qquad \text{and} \qquad \neg(A \vee B) = \neg A \wedge \neg B.$$

Divisibility

Lemma D.1 (Euclid's lemma) Let p, a, b be integers. If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

Lemma D.2 Let p, n be integers. If $p \mid n^2$, then $p \mid n$.

Sets

Lemma S.1 (De Morgan's Laws for Sets) Let A, B be sets. Then

$$(A \cap B)^c = A^c \cup B^c \qquad \text{and} \qquad (A \cup B)^c = A^c \cap B^c.$$

Lemma S.2 Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Lemma S.3 Let A, B be sets. Then the following statements hold.

(a) $A \cap B \subseteq A$

(b) $A \cup B \supseteq A$

(c) $A \setminus B = A \cap B^c$

(d) $(A^c)^c = A$

Functions

Lemma F.1 If $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is injective.

Lemma F.2 If $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f : A \to C$ is surjective.

Lemma F.3 A function is injective if and only if it has a left inverse.

Lemma F.4 A function is surjective if and only if it has a right inverse.

Lemma F.5 A function is bijective if and only if it has a two-sided inverse.

Cardinality

Lemma C.1 $\mathbb{N}$ is countable.

Lemma C.2 If I is countable and A_i is countable for all $i \in I$, then $\bigcup_{i \in I} A_i$ is countable.

Lemma C.3 If J is finite and A_j is countable for all $j \in J$, then $\prod_{j \in J} A_j$ is countable.

Lemma C.4 If $f : A \to B$ is injective and B is countable, then A is countable.

Lemma C.5 If $g : B \to A$ is surjective and B is countable, then A is countable.

Lemma C.6 $\mathbb{R}$ is uncountable.

Lemma C.7 All subsets of $\mathbb{R}$ that are intervals have the same cardinality as $\mathbb{R}$ itself.

Lemma C.8 (Schröder-Bernstein) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.